

# Multi-center MICZ-Kepler systems

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## Abstract

We present the classical solutions of the two-center MICZ-Kepler and MICZ-Kepler-Stark systems. Then we suggest the model of multi-center MICZ-Kepler system on the curved spaces equipped with  $so(3)$ -invariant conformal flat metrics.

## 1 Introduction

As is known, the Kepler system plays a special role among the finite-dimensional classical and quantum integrable systems. On the one hand, it has a number of physical applications, on the other hand, this system is one of the key components of hidden symmetry concept [1]. Various generalizations of this system were studied during last century. Among them are the Kepler problems on spheres and hyperboloids [2] as well as their higher dimensional counterparts. Further important generalization of the Kepler problem was constructed in the works of Zwanziger, and McIntosh and Cisneros [3]. They replaced the Coulomb center of the system by the Dirac dyon (electrically charged Dirac monopole [4]), and added to the initial Coulomb potential the specific centrifugal term  $s^2/2\mu r^2$ , where  $s = eg/c$  is the so-called monopole number ( $e$  is the electrical charge of the probe particle and  $g$  is the magnetic charge of monopole),  $\mu$  is a particle mass. As a result they obtained the integrable system which is very similar to the underlying Kepler one. It is presently known as a MICZ-Kepler system. For instance, MICZ-Kepler system inherent, besides the rotational symmetry, the hidden symmetry of the Kepler system given by the Runge-Lenz vector. Similar to the case of pure Coulomb system, the classical trajectories of the MICZ-Kepler system are second order curves. The quantum mechanical properties of Coulomb and MICZ-systems have also much in common. For example, the only difference in their energy spectra is the lift of the low bound of the orbital quantum number from 0 to  $|s|$ .

The MICZ-Kepler system describes the motion of the electrically charged scalar particle in the field of the Dirac dyons. The origin of the additional centrifugal term  $\frac{s^2}{2\mu r^2}$  can be also understood in this context. In the presence of monopole magnetic field the angular momentum of the system get additional, spin-like term,  $\mathbf{J} = \mathbf{r} \times \pi - \mathbf{s}\mathbf{r}/r$ , where  $\pi$  is dynamical momentum containing the vector potential of the monopole magnetic field. The magnetic field of Dirac dyon is  $\mathbf{B} = g\mathbf{r}/r^3$ . On the other hand, there is linear relation between orbital and magnetic momenta of single particle,  $\mathcal{M} = \frac{e}{2\mu c}\mathbf{J}$ . Hence, the interaction energy of this magnetic momentum with magnetic field  $\mathbf{B}$  is given by the expression:

$$U_B = -\mathcal{M}\mathbf{B} = -\frac{eg}{2\mu c}\mathbf{J}\frac{\mathbf{r}}{r^3} = \frac{s^2}{2\mu r^2}, \quad (1)$$

i.e. coincides with the centrifugal term in the MICZ-Kepler system [5]. Though, the additional part of the orbital momentum proportional to  $s$  should rather be assigned to the electro-magnetic field than to the particle, this non-correct interpretation nevertheless leads to the correct expression. Let us also mention, that MICZ-Kepler system also describes the relative motion of two Dirac dyons with electric and magnetic charges  $(e_1, g_1)$  and  $(e_2, g_2)$  with  $e_1g_2 - e_2g_1 = cs$ . Similar to above mentioned case, the centrifugal term  $s^2/2\mu r^2$  could be interpreted as the interaction energy of the induced dipole momentum with the dyon electric field plus interaction energy of the induced magnetic moment with magnetic field (here  $\mu$  is the reduced mass).

The analogy between MICZ-Kepler and Coulomb systems has elegant explanation in terms of four-dimensional space: both systems can be obtained from the Hamiltonian of four-dimensional oscillator via the reduction by  $U(1)$  group action [6]. In the same way, one can construct the five-dimensional analog of the MICZ-Kepler problem, reducing the eight-dimensional oscillator by  $SU(2)$  group action [7]. In this case, the  $SU(2)$  Yang monopole appears in the system instead of Dirac monopole. In two-dimensional case one can also construct MICZ-Kepler like system from the oscillator: for this purpose the two-dimensional (quantum) oscillator should be reduced by the action of the parity ( $Z_2$ ) group. The emerged two-dimensional MICZ-Kepler system is distinguished by the presence of magnetic flux which provides the system with spin  $1/2$  ( $Z_2$  monopole) [8]. The reduction procedure mentioned above has very deep and close relation with the Hopf

maps (see, e.g. [9]). For the detailed review of the quantum mechanical aspects of these issues the reader may see the Ref. [10]. The MICZ-Kepler systems on the three-dimensional spheres [11], on two-, three-, and five-dimensional two-sheet hyperboloids [12] as well as their generalization to arbitrary dimensions [13] are also distinguished by their close similarity to the corresponding Coulomb systems. For example, the only difference between the quantum-mechanical properties of Coulomb and the corresponding MICZ-Kepler systems consists in the lift of the range of the total angular momentum, which in its turn leads to the degeneracy of the ground state. Indeed, any three-dimensional rotationally invariant system (without monopoles) will preserve its main properties if incorporation of the Dirac monopole in the center will be supplied by the following change of potential

$$U(r) \rightarrow U(r) + \frac{s^2}{2Gr^2}, \quad (2)$$

where  $G(r)$  defines the conformal flat metric of the configuration space,  $ds^2 = G(r)(d\mathbf{r})^2$ . Particularly, the form of classical trajectory of the system,  $\varphi = \varphi(r)$  unaffected by this kind of replacement, whereas the orbital plane will not be orthogonal to the orbital momentum. Instead, the angle between them will be appeared which is defined by the expression  $\cos \theta = s/J$  [9]. The actual quantum-mechanical spectrum of the extended system coincides with that of the initial one, with minor modification of the possible range of orbital and magnetic quantum numbers  $j, m$ :  $j = |s|, |s|+1, \dots, m = -|s|, -|s|+1, \dots, |s|$  [14].

The problem of particle moving in the field of two Coulomb centers (or two-center Kepler problem) was solved in the middle of XIX century by Jacobi. He established the integrability of two-center Kepler system and of its limit when one of the forced centers is placed at infinity which yields the homogeneous potential field (we shall refer it as Kepler-Stark system) in elliptic and parabolic coordinates respectively. However, the generalization of this picture to case where Coulomb centers are replaced by dyons has been lacking until now. In our recent paper with S.Krivonos [15] we proposed the generalization of the MICZ-Kepler replacement (2) which can be used in case of  $N$  Dirac monopoles,

$$U(\mathbf{r}) \rightarrow \frac{1}{2G} \left( \frac{s_1}{r_1} + \dots + \frac{s_n}{r_n} \right)^2 + U(\mathbf{r}). \quad (3)$$

where  $s_i = eg_i$ , with  $g_i$  be the magnetic charge of the  $i$ -th monopole located at the point with coordinates  $\mathbf{a}_i$ , and  $r_i = |\mathbf{r} - \mathbf{a}_i|$ . This replacement has the following important features:

- The system (without monopoles) admitting separation of variables in elliptic/parabolic coordinates results in the separable two-center MICZ-system (3) with the Dirac monopoles placed at the foci of elliptic/parabolic coordinates.
- The system admits the  $\mathcal{N} = 4$  supersymmetric extension at the following choice of potential

$$U_0 = \frac{\kappa}{G} \left( \sum_I \frac{g_I}{|\mathbf{r} - \mathbf{a}_I|} + \mathbf{B}_0 \cdot \mathbf{r} \right) + \frac{\kappa^2}{2G}. \quad (4)$$

The corresponding supersymmetric system was constructed in earlier paper by Ivanov and Lechtenfeld [16].

On the Euclidean case the generalized MICZ-Kepler system yields the following generalization of the multi-center Kepler problem describing the motion of the particle with electric charge  $e$  in the field of  $N$  static Dirac dyons

$$\mathcal{H} = \frac{1}{2} \left( \mathbf{p} - \sum_{i=1}^N eg_i \mathbf{A}_D(\mathbf{r} - \mathbf{a}_i) \right)^2 + \frac{1}{2} \left( \sum_{i=1}^N \frac{eg_i}{|\mathbf{r} - \mathbf{a}_i|} \right)^2 + \sum_{i=1}^N \frac{eq_i}{|\mathbf{r} - \mathbf{a}_i|}, \quad (5)$$

where

$$\mathbf{A}_D(\mathbf{r}) = \frac{\mathbf{r} \times \mathbf{n}}{r(r - \mathbf{n}\mathbf{r})} \quad (6)$$

is the vector-potential of the Dirac monopole,  $g_i(q_i)$  are the its magnetic(electric) charge of  $i$ -th dyon, and  $\mathbf{n}_i$  is the unit vector pointed along the Dirac singularity line.

Since the two-center Kepler/Kepler-Stark system admit separation of variables in elliptic/parabolic coordinates, the expression mentioned above provides us with integrable two-center MICZ-Kepler system. Moreover, when the background dyons have the same ratio of the electric and magnetic charges (i.e. obey trivial Dirac-Schwinger-Zwanziger charge quantization condition), the potential of the system (5) belongs to the class (4), i.e. the system allows  $\mathcal{N} = 4$  supersymmetric extension. It is also noteworthy that the class of integrable models constructed in Ref. [17] include such two-center

MICZ-Kepler system in flat space as the limiting case. However, even classical trajectories of the suggested two-center MICZ-Kepler systems were not properly studied in the mentioned paper. Another disadvantage of the substitution (3) is that it is ill-defined for the MICZ-Kepler systems on spheres and hyperboloids. Namely, extending in this way the (one- and two-center) Kepler system on the sphere and hyperboloid one will obtain the system which á priori is not separable in the same coordinate systems as the initial one. Also these systems do not belong to the class allowing the  $\mathcal{N} = 4$  supersymmetric extensions.

In this paper we discuss these issues.

In the *Second Section* we describe in detail the two-center MICZ-Kepler and MICZ-Kepler-Stark systems and present their classical solutions.

In the *Third Section* we present the alternative model of the multi-center MICZ-Kepler system on three-dimensional sphere and hyperboloid, as well as on any  $SO(3)$  invariant spaces.

## 2 Integrability of two-center MICZ-Kepler system

Now let us demonstrate the classical integrability of the two-center MICZ-Kepler system, and construct the solutions. Suppose, that background dyons with magnetic(electric) charges  $g_{1,2}(q_{1,2})$  are fixed on  $z$ -axis at points  $(0, 0, -a)$  and  $(0, 0, a)$  respectively. Choosing the appropriate gauge for the vector-potentials in spherical coordinates

$$A_r^{1,2} = A_\theta^{1,2} = 0, \quad A_\varphi^{1,2} = g_{1,2} \cos \theta_{1,2}, \quad (7)$$

we arrive at the following Hamiltonian:

$$\mathcal{H} = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + \frac{(p_\varphi - s_1 \cos \theta_1 - s_2 \cos \theta_2)^2}{2r^2 \sin^2 \theta} + \frac{1}{2} \left( \frac{s_1}{r_1} + \frac{s_2}{r_2} \right)^2 + \frac{eq_1}{r_1} + \frac{eq_2}{r_2}. \quad (8)$$

In virtue of the relevant form of additional quadratic potential term the system allows separation of variables in elliptic coordinates given by the formulas:

$$\xi = \frac{r_1 + r_2}{2a}, \quad \eta = \frac{r_1 - r_2}{2a}, \quad \varphi = \varphi. \quad (9)$$

Using the general kinematic relations resulting from the geometry of problem

$$\begin{aligned} r_1 \sin \theta_1 &= r \sin \theta, & r_2 \sin \theta_2 &= r \sin \theta, \\ \cos(\theta_1 - \theta_2) &= \frac{4a^2 - r_1^2 - r_2^2}{2r_1 r_2}, \\ \cos \theta_1 &= \frac{r \cos \theta + a}{r_1}, & \cos \theta_2 &= \frac{r \cos \theta - a}{r_2}, \end{aligned} \quad (10)$$

one obtains the following expression for the Hamiltonian (8):

$$\mathcal{H} = \frac{1}{2a^2 (\xi^2 - \eta^2)} \left( (\xi^2 - 1)p_\xi^2 + (1 - \eta^2)p_\eta^2 + \mathcal{V}(\xi) + \mathcal{W}(\eta) \right), \quad (11)$$

where

$$\mathcal{V}(\xi) = \frac{p_\varphi^2 + s_+^2 - 2p_\varphi s_- \xi}{\xi^2 - 1} + 2aeq_+ \xi, \quad \mathcal{W}(\eta) = \frac{p_\varphi^2 + s_-^2 - 2p_\varphi s_+ \eta}{1 - \eta^2} + 2aeq_- \eta. \quad (12)$$

Here  $q_\pm = q_1 \pm q_2$ ,  $s_\pm = s_1 \pm s_2$ . Now choosing for the generating function the expression  $S = \varphi p_\varphi + S_1(\xi) + S_2(\eta) - Et$  we arrive at the following Hamilton-Jacobi equations:

$$(\xi^2 - 1) \left( \frac{dS_1}{d\xi} \right)^2 + \mathcal{V}(\xi) - 2a^2 E (\xi^2 - 1) = n, \quad (1 - \eta^2) \left( \frac{dS_2}{d\eta} \right)^2 + \mathcal{W}(\eta) - 2a^2 E (1 - \eta^2) = -n, \quad (13)$$

where  $n$  is the separation constant which fixes the values of the corresponding constant of motion

$$\mathcal{I}_e = -\frac{1}{\xi^2 - \eta^2} \left( \eta^2 ((\xi^2 - 1)p_\xi^2 + \mathcal{V}(\xi)) + \xi^2 ((1 - \eta^2)p_\eta^2 - \mathcal{W}(\eta)) \right) \quad (14)$$

Then, integrating Eqs. (13) one obtains the generating function of the system

$$S = p_\varphi \varphi - Et + \int \sqrt{2a^2 E + \frac{n - 2aeq + \xi}{\xi^2 - 1} - \frac{p_\varphi^2 + s_+^2 - 2p_\varphi s_- \xi}{(\xi^2 - 1)^2}} d\xi + \int \sqrt{2a^2 E - \frac{n + 2aeq - \eta}{1 - \eta^2} - \frac{p_\varphi^2 + s_-^2 - 2p_\varphi s_+ \eta}{(1 - \eta^2)^2}} d\eta \quad (15)$$

Hence, we proved the exact solvability of classical two-center MICZ-Kepler system. Now, taking derivatives of the generating function by constant of motion,  $\partial S/\partial p_\varphi$ ,  $\partial S/\partial n$  and  $\partial S/\partial E$  and putting them to constants (which we are claiming, without loss of generality, to be zero) one obtains the classical solution of the system. Its trajectories are defined by the expressions

$$\int \frac{d\xi}{(\xi^2 - 1)\sqrt{2a^2 E + \frac{n - \mathcal{V}(\xi)}{\xi^2 - 1}}} = \int \frac{d\eta}{(1 - \eta^2)\sqrt{2a^2 E - \frac{n + \mathcal{W}(\eta)}{1 - \eta^2}}} \quad (16)$$

$$\varphi = \int \frac{(p_\varphi - s_- \xi) d\xi}{(\xi^2 - 1)^2 \sqrt{2a^2 E + \frac{n - \mathcal{V}(\xi)}{\xi^2 - 1}}} + \int \frac{(p_\varphi - s_+ \eta) d\eta}{(1 - \eta^2)^2 \sqrt{2a^2 E - \frac{n + \mathcal{W}(\eta)}{1 - \eta^2}}}, \quad (17)$$

The time evolution of the system is given by the expression

$$t = a^2 \int \frac{d\xi}{\sqrt{2a^2 E + \frac{n - \mathcal{V}(\xi)}{\xi^2 - 1}}} + a^2 \int \frac{d\eta}{\sqrt{2a^2 E - \frac{n + \mathcal{W}(\eta)}{1 - \eta^2}}}. \quad (18)$$

Here  $t$  is the time and  $\mathcal{V}(\xi)$  and  $\mathcal{W}(\eta)$  are defined in Eqs. (12)

## 2.1 MICZ-Kepler-Stark(-Zeeman) system

The system introduced above has an important limiting case. Putting one of the dyons at the infinity one obtains the Hamiltonian of the charged particle moving in the field of one localized dyon with additional parallel homogeneous (constant uniform) electric and magnetic fields:

$$\mathcal{H} = \frac{(\mathbf{p} - e\mathbf{A}_D - e\mathbf{B} \times \mathbf{r}/2)^2}{2} + \frac{1}{2} \left( \frac{s}{r} + e\mathbf{B}\mathbf{r} \right)^2 + \frac{eq}{r} - e\mathbf{E}\mathbf{r}, \quad \mathbf{E} \parallel \mathbf{B} \quad (19)$$

Due to the presence of constant uniform magnetic and electric fields one could refer this system as MICZ-Kepler-Stark-Zeeman system. As in the previous case, here one have the integrable system, i. e. the system which allows separation of variables in parabolic coordinates.

In order to show it we, at first, turn to spherical coordinates in which the vector potential of the homogeneous magnetic field assumed to be in  $z$ -direction is

$$A_r = A_\theta = 0, \quad A_\varphi = \frac{1}{2} B r^2 \sin^2 \theta \quad (20)$$

thus, the Hamiltonian takes the form

$$\mathcal{H} = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + \frac{(p_\varphi - s \cos \theta - \frac{1}{2} e B r^2 \sin^2 \theta)^2}{2r^2 \sin^2 \theta} + \frac{1}{2} \left( \frac{s}{r} + e B z \right)^2 + \frac{eq}{r} - e E z \quad (21)$$

Due to additional quadratic term arising from the relevant MICZ-Kepler replacement the corresponding Hamilton-Jacobi equation become separable in parabolic coordinates given by the following relations:

$$\xi = r + z, \quad \eta = r - z \quad (22)$$

The Hamiltonian then reads

$$\mathcal{H} = \frac{1}{2(\xi + \eta)} (4\xi p_\xi^2 + 4\eta p_\eta^2 + \mathcal{V}(\xi) + \mathcal{W}(\eta)) - \frac{1}{2} p_\varphi e B, \quad (23)$$

where

$$\mathcal{V}(\xi) = \frac{(p_\varphi + s)^2}{\xi} + 3seB\xi - eE\xi^2 + \frac{e^2 B^2}{4} \xi^3 + 2 + eq, \quad \mathcal{W}(\eta) = \frac{(p_\varphi - s)^2}{\eta} - 3seB\eta + eE\eta^2 + \frac{e^2 B^2}{4} \eta^3 + 2eq. \quad (24)$$

Then, taking for the generating function  $S = \varphi p_\varphi + S_1(\xi) + S_2(\eta) - (E - \frac{1}{2}p_\varphi eB)t$  one arrive at the following Hamilton-Jacobi equations:

$$4\xi \left( \frac{dS_1}{d\xi} \right)^2 + \mathcal{V}(\xi) - 2E\xi = n, \quad 4\eta \left( \frac{dS_2}{d\eta} \right)^2 + \mathcal{W}(\eta) - 2E\eta = -n, \quad (25)$$

Here, as usual,  $n$  is the separation constant which fixes the values of specific constant of motion, responsible for the separation of variables. For this quantity from Eqs. (23) and (25) one obtains

$$\mathcal{I}_p = \frac{1}{\xi + \eta} (4\xi\eta(p_\xi^2 - p_\eta^2) + \eta\mathcal{V}(\xi) - \xi\mathcal{W}(\eta)) \quad (26)$$

The generating function obtained from the corresponding Hamilton-Jacobi equations is

$$S = \varphi p_\varphi - (E - \frac{1}{2}p_\varphi B)t + \frac{1}{2} \int \sqrt{2E - 3seB - \frac{(p_\varphi + s)^2}{\xi^2} + \frac{n - 2eq}{\xi} + eE\xi - \frac{e^2 B^2}{4}\xi^2} d\xi + \\ + \frac{1}{2} \int \sqrt{2E + 3seB - \frac{(p_\varphi - s)^2}{\eta^2} - \frac{n + 2eq}{\eta} - eE\eta - \frac{e^2 B^2}{4}\eta^2} d\eta. \quad (27)$$

Hence, we prove the classical integrability of the MICZ-Kepler-Stark-Zeeman system

From the generating function we immediately get the expressions for the trajectories of system

$$\int \frac{d\xi}{\xi \sqrt{2E + \frac{n - \mathcal{V}(\xi)}{\xi}}} = \int \frac{d\eta}{\eta \sqrt{2E - \frac{n + \mathcal{W}(\eta)}{\eta}}}, \quad (28)$$

$$\varphi = \frac{1}{2}Bt + \frac{1}{2} \int \frac{(p_\varphi + s)d\xi}{\xi \sqrt{2E + \frac{n - \mathcal{V}(\xi)}{\xi}}} + \frac{1}{2} \int \frac{(p_\varphi - s)d\eta}{\eta \sqrt{2E - \frac{n + \mathcal{W}(\eta)}{\eta}}}, \quad (29)$$

and for the time evolution

$$t = \int \frac{d\xi}{\sqrt{2E + \frac{n - \mathcal{V}(\xi)}{\xi}}} + \int \frac{d\eta}{\sqrt{2E - \frac{n + \mathcal{W}(\eta)}{\eta}}} \quad (30)$$

Here  $t$  is the time and  $\mathcal{V}(\xi)$  and  $\mathcal{W}(\eta)$  are defined in Eqs.(24)

### 3 Multi-center MICZ-Kepler system on curved space

As was mentioned in Introduction, any system (on conformal-flat space) without monopoles admitting the separation of variables in elliptic/parabolic coordinates remains separable in the same coordinate systems under incorporation of monopoles supplied by the corresponding modification of potential term given by (3). The two-center MICZ-Kepler systems considered in previous Section are particular examples of these systems. However, this procedure fails in case of the Coulomb systems on spheres and pseudospheres (two-sheet hyperboloids). Due to the existence of hidden symmetries given by Runge-Lenz vector the Coulomb system (on Euclidean space) admits the separation of variables in spherical, elliptic and parabolic coordinate. The Coulomb system on (pseudo)spheres also has the hidden symmetries given by the analog of Runge-Lenz vector. Analogously to Euclidean case, this hidden symmetry is connected to the separation variables in few coordinate systems. These coordinate systems turn to the spherical, elliptic and parabolic ones at the planar limit. The connection between spherical and Cartesian coordinates on the sphere is identical with that on the Euclidean space, whereas the discrepancy is appeared for the elliptic and parabolic ones (see, e.g., [18] and refs therein). Therefore, the statement of the previous Section concerning the separability of variables for “MICZ-extended” systems is no more valid for the Coulomb systems on (pseudo)sphere. Moreover, it seems that the substitution (3) is ill-defined on non-Euclidean spaces including (pseudo)spheres. The evidence of it stems out from the following. When the magnetic and electric charges of the background dyons obey trivial Dirac-Schwinger-Zwanziger quantization condition

$$g_i q_j - g_j q_i = 0, \quad (31)$$

the potential of the Euclidean multi-center MICZ-Kepler system belongs to the class (4) (up to unessential constant), i.e. it admits  $\mathcal{N} = 4$  supersymmetric extension. While the potential of the multi-center Coulomb system on (pseudo)sphere,

$$U(\mathbf{r}) = e\phi_{q_1, \dots, q_N} = e \sum_{i=1}^N q_i \phi(r_i), \quad \text{where} \quad \phi(r) = \frac{1}{2r_0} \frac{1 - \epsilon r^2}{r}, \quad r_i = |\mathbf{r} - \mathbf{a}_i| \quad (32)$$

does not belong to the class (4). Here  $\epsilon = 1$  correspond to the sphere, and  $\epsilon = -1$  to the pseudosphere, and the metrics looks as follows

$$ds^2 = \frac{4r_0^2 (d\mathbf{r})^2}{(1 + \epsilon r^2)^2}. \quad (33)$$

The (one-center) Coulomb potentials on Euclidean space, sphere and pseudosphere are nothing else, but the  $so(3)$  invariant Green functions for the Laplasians defined by the corresponding metrics,  $\Delta\phi_C = \delta(\mathbf{r})$ . While the vector potential of the Dirac monopole is just a one-form dual to this Green function. Explicitly,

$$*dA_D = -d\phi_C, \quad \Rightarrow \quad \Delta\phi_C = \delta(\mathbf{r}), \quad \Delta \equiv *d*d + d*d* \quad (34)$$

Taking into account the duality between vectors and one- and two-forms in the three-dimensional spaces, one can write down these expressions in the following way

$$-\nabla\phi_C = \text{rot}\mathbf{A}_D, \quad \Rightarrow \quad \Delta\phi_C = \delta(\mathbf{r}). \quad (35)$$

Let us write down the explicit  $so(3)$  invariant solutions of this equations on the  $so(3)$  invariant space with the metrics  $ds^2 = G(r)(d\mathbf{r})^2$ . The (zero-, one and two-)form fields are independent on the metric, hence, one has to choose

$$d\mathbf{A}_D(\mathbf{r}) = \frac{(\mathbf{r} \times d\mathbf{r}) \wedge d\mathbf{r}}{2r^3}, \quad d\phi(r) = \frac{d\phi(r)}{dr} \frac{\mathbf{r} d\mathbf{r}}{r}. \quad (36)$$

Taking into account the conformal flatness of the metrics, we get, from (34)

$$\frac{d\phi_C}{dr} = -\frac{1}{r^2 \sqrt{G(r)}}, \quad \Rightarrow \quad \phi_C = -\int \frac{dr}{r^2 \sqrt{G(r)}}. \quad (37)$$

Particularly, for the Euclidean space one obtains  $\phi_C = 1/r$ , and for the (pseudo)sphere (33) the potential given in (32).

Now, we define the multi-center MICZ-Kepler system on the  $so(3)$  invariant space by the expression

$$\mathcal{H} = \frac{(\mathbf{p} - e\mathbf{A}_{g_1, \dots, g_N})^2}{2G} + \frac{e^2 \phi_{g_1, \dots, g_N}^2}{2} + e\phi_{q_1, \dots, q_N}, \quad (38)$$

where  $\mathbf{A}_{g_1, \dots, g_N} = \sum_i g_i \mathbf{A}_D(\mathbf{r} - \mathbf{a}_i)$ ,  $\phi_{q_1, \dots, q_N} = \sum_i q_i \phi_C(r_i)$ .

The corresponding one-center Hamiltonian on the sphere and two-sheet hyperboloid coincides with the MICZ-Kepler Hamiltonian constructed within the rule (2) up to unessential constant  $\epsilon s^2/4r_0^2$ . Hence, it is separable in the spherical coordinates, and in the modified elliptic and parabolic coordinates considered in Ref. [18]. It seems, that two-center MICZ-Kepler systems on spheres and pseudospheres are also separable in these coordinates.

When the electric and magnetic charges of background dyons obey the condition (31), or, equivalently,  $q_i/g_i = \epsilon\kappa$ , this Hamiltonian can be represented in the form

$$\mathcal{H} = \frac{(\mathbf{p} - e\mathbf{A}_{g_1, \dots, g_N})^2}{2G} + \frac{e^2(\phi_{g_1, \dots, g_N} + \kappa/e)^2}{2} - \frac{\kappa^2}{2}. \quad (39)$$

where  $\text{grad}(\phi_{g_1, \dots, g_N} + \kappa/e) = -\text{rot}\mathbf{A}_{g_1, \dots, g_N}$ . This Hamiltonian admits the  $\mathcal{N} = 4$  supersymmetric extension on the Euclidean space ([16]) and on the sphere ([19]) (provided the unessential constant  $\kappa^2/2$  is omitted). In our knowledge, the supersymmetric extensions of this systems on the generic conformal-flat space are unknown. However, we believe, that with the explicit component expressions for the supercharges of the system on sphere [?], we will be able to guess the supersymmetric system on any conformal flat case.

**Acknowledgements.** We are indebted to Sergey Krivonos for collaboration at the earlier stage of the work and drawing our attention to his paper on supersymmetric mechanics with monopole on sphere [19], which allowed us to suggest the MICZ-Kepler system on curved spaces given by Hamiltonian (38). We are grateful to George Pogosyan for pointing out us the fact that the Coulomb potential on the sphere is the Green function of the corresponding Laplace operator, to Tigran Hakobyan and Vagharshak Mkhitarian for discussions and the interest in work, and to the Organizers of Conference on Classical and Quantum Integrable Systems for kind invitation and hospitality in Dubna. This work was supported in part by the grants NFSAT-CRDF ARP1-3228-YE-04 and INTAS-05-7928.

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